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Localization in GWZW and Verlinde formula

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ABSTRACT

Gauged Wess-Zumino-Witten theory for compact groups is considered. It is shown that this theory has fermionic BRST-like symmetry and may be exactly solved using localization approach. As an example we calculate functional integral for the case of $SU(2)$ group on the arbitrary Riemann surface. The answer is the particular case of Verlinde formula for the number of conformal blocks.

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1 Introduction

One of the interesting problem in Quantum Field Theory is to understand in unifying terms the nature of integrability of various kinds of known quantum exactly soluble theories. It seems that quantum exactly soluble theories are in the class of the theories in which after possible addition of auxiliary fields we may use for all but finite number of modes quasiclassical approximation to get exact answers. Thus in these theories it is possible to reduce explicitly all computations to finite dimensional integral (or even to the number of one-dimensional integrals) and to the problem to find solutions of the classical equations of motion. This is apparently implies some kind of fixed point formula. It is quite desirable to find the universal method to solve these theories and probably the best candidate is cohomological approach developed in [2, 1] and recently reconsidered in [3, 4, 5, 6].

We know several classes of two-dimensional theories to which localization or cohomological approach is (or probably may be) applicable. The most natural class consists of topological theories with explicit BRST symmetry and the correlation functions of these theories may be described in terms of topological characteristics of some finite dimensional geometrical objects.

We have also trivial class of exactly soluble theories . It is free theories with quadratic action where quasiclassical approximation is exact. Conformal field theories give a slightly more complex example of exactly soluble theories closely connected with free massless theories. The difference is in additional constraints on finite number of degrees of freedom. This suggests that correlation functions of any conformal theory on any Riemann surface may be calculated directly by some version of localization methods. When coupled to gravity these theories are deeply related with topological theories and surely admit localization.

It is interesting to consider general 2d integrable theories from this point of view. Taking into account possible auxiliary fields these theories may be treated as deformation of conformal theories which somehow preserves "free field" description but now these fields are massive (see e.x.[15]). Here also different nontrivial classical solutions are possible and the whole picture is more complex but the existence of good action-angle variables encouraged to try to apply localization approach. Surely it is natural to begin with finite-dimensional integrable systems using the Lagrangian description [15] and the explicit quantization [7]

Notice that there is also general construction of BRST-like symmetry in Hamiltonian formalism for Hamiltonian with rather mild restrictions which formally gives the credit to quasiclassical approximation to be exact [4, 5, 6] but exact conditions of applicability of this formalism are unclear.

Nice example of application of localization method is given by 2d Yang-Mills (YM) theory ([9, 10]). Here it is possible to reproduce the known answers for correlation functions [8] and to connect YM theory with topological theory which calculates volume of moduli space of flat connections.

Bellow we will consider slightly more complex example of Gauged-Wess-Zumino-Witten (GWZW)[11]. It is topological two-dimensional theory closely related with conformal Wess-Zumino-Witten model (WZW). In particular its functional integral gives the number of conformal blocks in WZW. Elegant formula due to Verlinde [12] is known for the number of conformal blocks. We will show that the functional integral may be exactly calculated and reproduces the Verlinde formula. As an example we will consider the case of $SU(2)$. It is interesting to remark that in this calculation we should treat finite number (one in the case of $SU(2)$) degrees of freedom exactly and the rest part quasiclassicaly.

The content is the following . In the second section we describe simple derivation of the action of GWZW model with the emphasis on the connection with 3d Chern-Simons theory [13]. Section 3 is devoted to reconsideration of 2d Yang-Mills theory where we pay attention on the one-loop corrections to classical answer. In section 4 we describe the calculation of functional integral for GWZW with group $SU(2)$ and obtain Verlinde formula for the number of conformal blocks.

2 Gauged Wess - Zumino Theory (GWZW)

In this section we will give a short derivation of functional integral representation for GWZW theory starting with Chern-Simons (CS) theory. This is revealed the connection of correlation functions of GWZW and correlation functions of CS theory which in turn are connected with the modular geometry of conformal blocks of WZW. In particular it will be obvious that vacuum correlation functions of GWZW are equal to the dimension of the space of conformal blocks in Wess-Zumino-Witten model.

Consider CS theory with compact gauge group G on the three dimensional space M with the topology of the product of the one-dimensional circle and two-dimensional Riemann surface $M = S^1 \times \Sigma$. Let us given the connection A with the curvature F and some integer k.Then using explicit (2+1) notations functional integral for CS-theory may be represented in the form:

$$Z = \int \mathcal{D}A \, e^{k \int (A_i \partial_t A_j \epsilon^{ij} + \int A_0 F_{i,j}(A) \epsilon^{ij}) dt d^2 x} \tag{1}$$

where 0-component is along S^1 and x^i are coordinates on Σ. It is known ([13]) that Hilbert space of CS theory is given by the cohomology groups of the moduli space $\mathcal{M}^G$ of flat connections with gauge group G with coefficients in the k^{th} power of the determinant bundle.

$$\mathcal{H}_{CS} = H^0(\mathcal{M}^G; \mathcal{L}^{\otimes k}) \tag{2}$$

(under the condition the higher cohomology groups are zero). On the other side the elements of this cohomology groups are corresponded to conformal blocks in the WZW theory with the Kac-Moody group $\widehat{G}_k$. CS theory has zero Hamiltonian and the partition function of CS-theory on M may be interpreted as the trace of unit operator over the space of conformal blocks and hence equal to the dimension of the space :

$$Z = dim \mathcal{H}_{CS} = dim H^0(\mathcal{M}^G; \mathcal{L}^{\otimes k}) \tag{3}$$

Usually the higher cohomology are zero and the dimension of the zero cohomology group may be calculated through Riemann-Roch theorem:

$$\sum_p (-1)^p dim H^p(\mathcal{M}^G; \mathcal{L}^{\otimes k}) = \int_{\mathcal{M}^G} Ch(\mathcal{L}) Td(T\mathcal{M}^G) \tag{4}$$

where Chern character and Todd class in the case of direct sum of linear bundles $E = \oplus x_i$ are :

$$Ch(E) = \sum_i e^{x_i} \tag{5}$$

$$Td(E) = \prod_i \frac{x_i}{1 - e^{-x_i}} \tag{6}$$

It is obvious that the role of S^1 is rather trivial in the above considerations and thus we may suppose that the same answer is obtained in the 2d theory which appears when the radius of the S^1 goes to zero . Bellow we will show that along this line we will obtain GWZW theory .

To get the partition function on the $M = S^1 \times \Sigma$ we will consider the propagator K in CS theory. From the structure of the action of CS theory we could easily derive that the propagator may be represented as a product of the propagator in the theory defined by the functional integral:

$$Z_{FreeCS} = \int \mathcal{D}A e^{k \int A_i \partial_t A_j \epsilon^{i,j}} \tag{7}$$

and the projector on the states with zero two-dimensional curvature. To construct both peaces let us define the scalar product on the space of wave functions over space of all connections :

$$< \Psi_1 | \Psi_2 >= \int \mathcal{D}A_z \mathcal{D}A_{\overline{z}} e^{(-k \int A_z A_{\overline{z}})} \Psi\left(A_z\right) \overline{\Psi(A_{\overline{z}})} \tag{8}$$

where $\Psi\left(A_z\right)$ is wave function. For such scalar product propagator in the theory without constraints is :

$$K_{FreeCS}(A_z^1, A_{\overline{z}}^2) = e^{-k \int A_z^1 A_{\overline{z}}^2} \tag{9}$$

For example we may verify that

$$K_{FreeCS}^2 = K_{FreeCS} \tag{10}$$

as it should be for the theory with zero Hamiltonian.

Constraint $F(A_z, A_{\overline{z}}) = 0$ for the polarization when wave function depends only on A_z is:

$$\left(\partial \frac{\delta}{\delta A} - \overline{\partial} A + \left[A, \frac{\delta}{\delta A} \right] \right) \Psi(A) = 0 \tag{11}$$

This condition may be interpreted as infinitesimal variant of the following condition of "gauge invariance" of wave functions of CS -theory:

$$\Psi\left(A^g\right) = e^{k S_{WZW}(g) + k \int A g^{-1} \overline{\partial} g} \Psi\left(A\right) \tag{12}$$

$$A^g = g^{-1} dg + g^{-1} A g \tag{13}$$

where S_{WZW} is the standard action of WZW theory. Using Polyakov-Wiegman formula it is simple to verify that the following projector operator is compatible with the condition (2).

$$\Pi\Psi(A) = \int \mathcal{D}g e^{kS_{WZW}(g)+k\int Ag^{-1}\overline{\partial}g}\Psi\left(A^g\right) \tag{14}$$

We normalize the projector to have the necessary property for any projector:

$$\Pi^2 = \Pi \tag{15}$$

Now we may calculate the partition function of CS theory :

$$\begin{aligned} Z = TrK_{FreeCS}\Pi = \int \mathcal{D}A_z\mathcal{D}A_{\overline{z}}\mathcal{D}g e^{-k\int A_zA_{\overline{z}}+kS_{WZW}(g)} \\ e^{\int Ag^{-1}\overline{\partial}g+k\int A_zA_{\overline{z}}^g} = \int \mathcal{D}A_z\mathcal{D}A_{\overline{z}}\mathcal{D}g e^{kS_{GWZW}(g,A_z,A_{\overline{z}})} \end{aligned} \tag{16}$$

This is the functional integral for Gauged Wess-Zumino-Witten theory with the action:

$$\begin{aligned} S_{GWZW}(g,A) = kS_{WZW}(g) + \\ \frac{i}{2\pi}k\left(\int Ag^{-1}\overline{\partial}g + g\partial g^{-1}A_{\overline{z}} + gA_zg^{-1}A_{\overline{z}} - A_zA_{\overline{z}}\right) \end{aligned} \tag{17}$$

Let us mention another interesting representation for the GWZW as the ratio of two determinants. Taking into account the expression for regularized determinant:

$$det\Delta\left[A\right] = e^{S_{WZW}[A]+S_{WZW}\left[\overline{A}\right]+\int A\overline{A}} \tag{18}$$

we have the following representation:

$$e^{S_{GWZW}(g,A_z,A_{\overline{z}})} = \frac{det\Delta\left[A,\overline{A}^g\right]}{det\Delta\left[A,\overline{A}\right]} \tag{19}$$

$$\overline{A}^g = g^{-1}\overline{\partial}g + g^{-1}\overline{A}g \tag{20}$$

To summarize we have 2d theory with the property that its functional integral gives the number of conformal blocks in WZW theory. It appears that for this quantity there exist rather simple representation. Using the nontrivial fact that structure constants of fusion algebra N_{ij}^k:

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k \tag{21}$$

are diagonalized by the matrix of modular transformation of one loop conformal blocks Verlinde ([12]) deduce formula for the number of conformal blocks of arbitrary conformal theory on the surface Σ of genus g through the matrix of one-loop modular transformation S_{ij}:

$$dimV_g = tr\left(\sum_{i=0}(N^i)^2\right)^{g-1} = tr\left(\sum_{i=0}\frac{(S_i^h)^2}{(S_0^h)^2}\right) = \sum_{n=0}^{N-1}|S_{n0}|^{2(1-g)} \tag{22}$$

Im particular case of WZW with the group $\widehat{G}_k$ it has the form:

$$dimV_\Sigma^g = (C(k+h)^r)^{g-1}\sum_{\lambda\in P_+^k}\prod_{\alpha\in\Delta}\left(1-e^{2\pi i\alpha\left(\frac{\lambda+\rho}{k+h}\right)}\right)^{1-g} \tag{23}$$

where C is the order of the center and h - Coxeter number, ρ is the half of the sum of positive roots and P_+^k - weights of integrable representations of $\widehat{G}_k$. Below we derive this formula for the case of $\widehat{SU}(2)_k$ on arbitrary surface. In this case the number of conformal blocks is given by

$$dimV_\Sigma^g = \sum_{n=0}^{N-1}|S_{n0}|^{2(1-g)} = \sum_{n=0}^{k+1}\left(\frac{k+2}{2}\right)^{g-1}\left(sin\left(\frac{n+1}{k+2}\pi\right)\right)^{2(1-g)} \tag{24}$$

Bellow we will show how this formula may be derived from functional integral (16)

3 Two-dimensional Yang-Mills theory

In this section we reconsider the calculation of functional integral for 2d Yang-Mills theory described in [9, 10]. The difference with [9, 10] will be in taking into account non-trivial one-loop contribution which provide us with the exact result.

Let A be the connection on the surface Σ with the curvature $F = dA + \frac{1}{2}[A, A]$ and ϕ is the scalar field in adjoint representation. Then the action of two-dimensional Yang-Mills theory may be written as:

$$S = tr \int (\phi F + \frac{1}{2}\psi \wedge \psi + \epsilon\phi^2) \tag{25}$$

where we introduce the fermions ψ to have a canonical measure $dAd\psi$

This theory posses the following BRST-like symmetry:

$$\delta A_\mu = \lambda\psi_\mu \tag{26}$$
$$\delta\psi_\mu = \lambda D_\mu\phi = \tag{27}$$
$$\lambda(\partial_\mu\phi + [A_\mu, \phi]]) \tag{28}$$
$$\delta\phi = 0 \tag{29}$$

We may define an invariant observables by the condition that if the observable is represented as the integral of some operator-valued form over closed cycle the BRST variation of this form should be exact [3]:

$$\left[Q, \mathcal{O}_n^{(0)}\right] = 0 \tag{30}$$
$$\left[Q, \mathcal{O}_n^{(1)}\right] = d\mathcal{O}_n^{(0)} \tag{31}$$
$$\left[Q, \mathcal{O}_n^{(2)}\right] = d\mathcal{O}_n^{(1)} \tag{32}$$
$$\tag{33}$$

From (3) we have obvious zero-form observables $\mathcal{O}_n^{(0)} = tr\phi^n$ and from (33) we find the whole family of observables:

$$\mathcal{O}_n^{(0)} = tr\phi^n \tag{34}$$
$$\mathcal{O}_n^{(1)} = ntr\phi^{n-1}\psi \tag{35}$$
$$\mathcal{O}_n^{(2)} = ntr\phi^{n-1}F + \frac{n}{2}tr\sum_{k=0}^{n-2}\phi^k\psi \wedge \phi^{n-k-2}\psi \tag{36}$$
$$\tag{37}$$

To calculate the functional integral which respect some fermionic symmetry we should find such fermionic function V that deformed action:

$$S = S_0 + \beta\,[Q, V] \tag{38}$$

with $\beta \to \infty$ gives rise to concentration of the functional integral on some small finite dimensional subspace of the field configurations. The useful choice in our situation is;

$$\delta S = [Q, V] \tag{39}$$

$$V = \beta \int \Psi(\bigtriangledown\phi - \bigtriangledown F) \tag{40}$$

$$\delta S = \beta \int \bigtriangledown\phi(\bigtriangledown\phi - \bigtriangledown F) + \phi\Psi \wedge \Psi + \Psi\Delta\Psi + F\Psi \wedge \Psi = \tag{41}$$

$$\beta \int (\bigtriangledown\phi - \bigtriangledown F)^2 - |\bigtriangledown F|^2 + (\phi - F)\Psi \wedge \Psi + \Psi\Delta\Psi \tag{42}$$

Let us note that by very general arguments [14] functional integral with fermionic symmetry concentrates in infinitesimal neighborhood of the zero locus of symmetry transformations. Thus we may suppose from explicit form (3) that quasiclassical approximation is exact for all modes with exception of zero modes of the field ϕ which should be treated exactly. This is in agreement with the (3). It is useful to divide the field ϕ on constant and nonconstant parts:

$$\phi = \phi_0 + \phi_* \tag{43}$$

Then bosonic part of the action has the form:

$$\int (\phi F + \epsilon\phi^2) = \int ((\phi_0 + \phi_*)F + \epsilon\phi_0^2 + \epsilon\phi_*^2) \tag{44}$$

Classical equations of motions are :

$$\begin{aligned} \Psi &= 0 \\ \bigtriangledown\phi_* &= 0 \\ F_* + 2\epsilon\phi_* &= 0 \end{aligned} \tag{45}$$

As the consequence we have:

$$\bigtriangledown F = 0 \tag{46}$$

With the appropriate gauge transformation in $SU(2)$ case we may choose representative which has only one non-zero component of ϕ along σ_3 direction (for example). From the condition on ϕ to be covariantly constant we deduce that ϕ is constant ($\phi_* = 0$) and ($\pm$)- components (with respect to decomposition of $sl(2)$ on lower-triangle (+) ,diagonal and upper-triangle (-) parts) of gauge fields are zero. Hence we are reduced to $U(1)$-bundle on the surface and from topological restrictions the values of the curvature (which is constant) is $2\pi m$ where m should be integer.

Now consider the second variation of the action.

$$\delta^2 S = tr \int 2\phi^0 \left[\delta A_\mu, \delta A_\nu\right] + 2\delta\phi^+ (\bigtriangledown_\mu \delta A_\nu)^- + 2\delta\phi^- (\bigtriangledown_\mu \delta A_\nu)^+ + \tag{47}$$

$$\delta\phi^0 \partial_\mu \delta A^0_\nu + 2\epsilon\delta\phi^0\phi^0 + 2\epsilon\delta\phi^+\delta\phi^- + 2\epsilon\delta\phi^-\phi^+ = \tag{48}$$

$$= tr \int \phi^0 \left(\delta A^+_\mu - \bigtriangledown_\mu \left(+\frac{\delta\phi^+}{2\phi^0}\right)\right)\left(\delta A^-_\mu - \bigtriangledown_\mu \left(-\frac{\delta\phi^-}{2\phi^0}\right)\right) + \tag{49}$$

$$+\delta\phi^0 \partial_\mu \delta A_\nu 2\epsilon\delta\phi^0\phi^0 \tag{50}$$

By shifting integration over connections we get the following representation for the action up to second order around the classical solution parametrized by integer $m = \frac{1}{2\pi}\int F$ and constant mode of the field ϕ :

$$S = \int \phi^0_0 2\pi i m + \int \epsilon(\phi^0_0)^2 + \int \phi_0 \delta A^+_\mu \delta A^-_\nu + \epsilon(\delta\phi^0_*)^2 + \delta\phi^0_* \partial_\mu \delta A^0_\nu \tag{51}$$

Taking into account that the action is invariant under the gauge symmetry we may parametrize $\phi^\pm$ in terms of infinitesimal gauge parameterλ:

$$\delta\phi^\pm = \left[\lambda^\pm, \phi^0\right] \tag{52}$$

The space of the classical solutions is reduced in essence to the moduli space of $U(1)$ connections and the measure in the vicinity of the classical solution may be represented in the form:

$$d\phi^+ d\phi^- d\phi^0 dA^+ dA^- dA^0 d\Psi^+ d\Psi^- d\Psi^0 = \tag{53}$$

$$d\lambda^+ d\lambda^- d\lambda^0 \left[\frac{dA^+ dA^- dA^0}{d\lambda^0}\right] d\mu_J det \left[2\phi^0 \mathbf{I}\right]^2 \tag{54}$$

In the previous expression $d\mu_J$ is the natural measure on the Jacobian of the curve and $\mathbf{I}$ is the unit operator. Taking Gauss integral we get the following "quasiclassical" answer in the topological sector parametrized by integer m:

$$Z_m = e^{S_{class}} Vol_{\text{Jacobian}} Vol_{\text{Gauge group}} \frac{det\left[2\phi^0\mathbf{I}\right]^{2(g-1)}_{(0)}}{det\left[2\phi^0\mathbf{I}\right]^{2(g-1)}_{(1)}} \tag{55}$$

where $det_{0,1}$ are determinants over the space of 0(1) differentials. As usual we should divide on the volume of the gauge group $Vol_{\text{Gauge group}}$. Let us compare accurately the number of eigenvalues of the 1-differential and 0-differential eigenvalues. Consider the differential operator

$$\overline{D} = \overline{\partial} + \overline{A} : \Omega^0 \to \Omega^1 \tag{56}$$

Then it is natural to define the difference to be equal to:

$$\text{Dim Ker } \overline{D} - \text{Dim CoKer } \overline{D} = \left[\begin{array}{c} \text{for (+) subalgebra } g - 1 + \int F \\ \text{for (-) subalgebra } g - 1 - \int F \end{array} \right. \tag{57}$$

Thus we have:

$$\phi_0^{2(g-1)+\int F - \int F} = \phi_0^{2(g-1)} \tag{58}$$

and the ratio of determinants is equal to:

$$\frac{det\,[2\phi^0 \mathbf{I}]_{(0)}^{2(g-1)}}{det\,[2\phi^0 \mathbf{I}]_{(1)}^{2(g-1)}} = \frac{1}{\phi^0}^{2(g-1)} \tag{59}$$

The final result is given by the sum over all topologically different classes of classical solutions. It is easy to see that in this case we obtain the following expression:

$$Z = \sum_m \int dy e^{2my - \epsilon y^2} \left(\frac{1}{y}\right)^{2(g-1)} = \sum_n e^{\epsilon n^2} \left(\frac{1}{n}\right)^{2(g-1)} \tag{60}$$

It is simple to show that all above calculations may be fulfilled in slightly more general case of the arbitrary potential for the field ϕ

$$S = tr(\int \phi F + \sum_k \epsilon_k \phi^k) \tag{61}$$

In this case we get the following natural generalization of (3)

$$S = \sum_m e^{im\phi y + \sum_k \epsilon_k y^k} (\frac{1}{y})^{2(g-1)} = \sum_n e^{\sum_k \epsilon_k n^k} (\frac{1}{n})^{2(g-1)} \tag{62}$$

4 GWZW theory

Now we consider functional integral for two-dimensional theory with the action of the form:

$$S = kS_{WZW}(g) + \frac{ik}{2\pi}\int (A_z g^{-1}\overline{\partial} g + g\partial g^{-1} A_{\overline{z}} + gA_z g^{-1} A_{\overline{z}} - A_z A_{\overline{z}}) \quad (63)$$

Notice that these action is invariant under transformation:

$$g \to hgh^{-1} \quad (64)$$
$$A_{\overline{z}} \to hA_{\overline{z}}h^{-1} + h\overline{\partial}h^{-1} \quad (65)$$
$$A_z \to hA_{\overline{z}}h^{-1} + h\partial h^{-1} \quad (66)$$

It will be useful to deal with slightly deformed theory:

$$S = S_{GWZW} + \epsilon Tr\int d^2z(g-1) \quad (67)$$

which preserve all the symmetries of the action and functional integral with this action may be considered as generating function of the correlators in GWZW .

This theory is connected with YM theory considered in the previous section. Introducing the parametrization:

$$g = e^{\frac{i\phi}{k}} \quad (68)$$
$$\epsilon = k^2\widetilde{\epsilon} \quad (69)$$

we get YM theory from GWZW when k goes to ∞:

$$S_{k\to\infty} = tr\int \phi F + \widetilde{\epsilon} tr\int \phi^2 \quad (70)$$

It appears that if we write the measure for A as an integral over fermions $\Psi_z, \Psi_{\overline{z}}$ similar to the case of YM theory GWZW theory will exhibit BRST-like symmetry with fermionic parameter λ:

$$\delta A_\mu = \lambda \Psi_\mu \quad (71)$$
$$\delta \Psi_{\overline{z}} = \lambda(A^g_{\overline{z}} - A_{\overline{z}}) \quad (72)$$
$$\delta \Psi_{\overline{z}} = -\lambda(A^{g^{-1}}_z - A_z) \quad (73)$$
$$\delta g = 0 \quad (74)$$

To calculate exactly functional integral in GWZW we may deform the action in the following way:

$$\delta S = [Q, V] \tag{75}$$

$$V = \beta \int \Psi(A^g_{\overline{z}} - A_{\overline{z}}) + \overline{\Psi}(A^{g^{-1}}_z - A_z) + \Psi \bigtriangledown_{A,\overline{A}^g} F(A, \overline{A}^g) \tag{76}$$

$$\delta S = \beta \int |A^{g^{-1}}_z - A_z|^2 + |A^g_{\overline{z}} - A_{\overline{z}}|^2 + (A^g_{\overline{z}} - A_{\overline{z}}) \bigtriangledown F + \tag{77}$$

$$(A^{g^{-1}}_z - A_z)\overline{\bigtriangledown} f \Psi(g^{-1}\overline{\Psi}g - \overline{\Psi}) + \tag{78}$$

$$\overline{\Psi}(g\Psi g^{-1} - \Psi) + \Psi\delta(\overline{\bigtriangledown}F) + \overline{\Psi}\delta(\bigtriangledown F) \tag{79}$$

or just look at the zero locus of BRST transformations. In any case we derive that this functional integral may be localized on the solutions of of equations of motions with the exception of zero mode of the field g which should be treated exactly. Consider equation of motions for GWZW :

$$g^{-1}\overline{\partial}g + g^{-1}A_{\overline{z}}g - A_{\overline{z}} = 0 \tag{80}$$

$$g\overline{\partial}g^{-1} + gA_{\overline{z}}g^{-1} - A_{\overline{z}} = 0 \tag{81}$$

$$F_{z,\overline{z}}(A_z, g^{-1}\overline{\partial}g + g^{-1}A_{\overline{z}}g) = 0 \tag{82}$$

We will restrict ourselves to the case of $SU(2)$. From the condition for g to be covariantly constant by usual arguments we get that we may reduce g to be constant matrix in σ_3 subgroup of $SU(2)$.

We parametrize the element of the group as;

$$g = e^{i\phi\sigma_3} = cos\phi + i\sigma_3 sin\phi \tag{83}$$

and the classical action will have the form:

$$S = k\int F\phi + \epsilon cos\phi \tag{84}$$

Taking into account the topological restrictions on total curvature on the surface we have for particular topological sector :

$$S^{(m)}_k = 2\pi ikm\phi + \epsilon cos\phi \tag{85}$$

The second variation of the action is given by the expression:

$$\delta^2 S = k\int(\delta A_z g^{-1}\delta A_{\overline{z}}g - \delta A_z\delta A_{\overline{z}}) + \int \bigtriangledown_A(g^{-1}\delta g)\overline{\bigtriangledown}_{A^g}(g^{-1}\delta g) \tag{86}$$

$$+\int \delta A\overline{\bigtriangledown}_{A^g}(g^{-1}\delta g) + \int g^{-1}\delta A_{\overline{z}}g \bigtriangledown_A (g^{-1}\delta g) + trgg^{-1}\delta g g^{-1}\delta g \tag{87}$$

It is useful to parametrize g in terms of gauge transformations $h = e^{\omega} = 1 + \omega + \ldots$:

$$\delta g = \delta(h^{-1}e^{\phi\sigma_3}h) = \delta\phi e^{\phi\sigma_3} + e^{\phi}\omega - \omega e^{\phi} \tag{88}$$
$$(g^{-1}\delta g)^+ = (1 - e^{2\phi})\omega^+ \tag{89}$$
$$(g^{-1}\delta g)^- = (1 - e^{-2\phi})\omega^- \tag{90}$$
$$(g^{-1}\delta g)^0 = \delta\phi^0 \tag{91}$$

In this parametrization the second variation of the action will be :

$$\partial\delta\phi^0\overline{\partial}\delta\phi^0 + 2\delta\overline{A}^0\partial\delta^0 + 2\delta A^0\overline{\partial}\phi^0 + \tag{92}$$
$$\left(e^{2\phi} - 1\right)(\delta\overline{A}^+ + \overline{\bigtriangledown}\omega^+)(\delta\overline{A}^- + \overline{\bigtriangledown}\omega^-) + \left(e^{-2\phi} - 1\right)(\delta\overline{A}^- \tag{93}$$
$$+\overline{\bigtriangledown}\omega^-)(\delta\overline{A}^+ + \overline{\bigtriangledown}\omega^+) \tag{94}$$

The difference with YM case is mainly in one-loop contribution which for GWZW has the following form (we take into account ghost contribution) :

$$\frac{det_0(e^{2\phi} - 1)(e^{-2\phi} - 1)\mathbf{I}}{det_{1^+}((e^{2\phi} - 1)\mathbf{I})det_{1^-}((e^{-2\phi} - 1)\mathbf{I})} \tag{95}$$

where $det_0(\mathbf{I})$ is determinant of the unit operator on the space of 0-forms,and $det_{1^\pm}(\mathbf{I})$ is the determinant on the space of one-forms proportional to $\sigma^\pm$. Using the same arguments as in the case of YM theory we have for the ratio of the determinants:

$$(e^{2\phi} - 1)^{g-1+\int F}(e^{-2\phi} - 1)^{g-1-\int F} = \tag{96}$$
$$= \frac{1}{(2sin\phi)^{2(g-1)}}e^{c_v\int F\phi} \tag{97}$$

where dual Coxeter number is $c_v = 2$ for $SU(2)$.

Thus we get the well known renormalisation:

$$k \to k + c_v \tag{98}$$

The final step of calculation looks as following:

$$Z(\epsilon) = \sum_m \int e^{(k+2)m\phi + \epsilon(cos\phi - 1)} \frac{2^{g-1}(k+2)^{g-1}}{(2sin\phi)^{2(g-1)}} d\phi = \tag{99}$$
$$= \sum_{l=1}^{k+1} e^{\epsilon(cos\frac{2\pi l}{k+2} - 1)} \frac{2^{g-1}(k+2)^{g-1}}{(sin\frac{2\pi l}{k+2})^{2(g-1)}} \tag{100}$$

where $(k+2)^{g-1}$ is the volume of the Jacobian divided on normalized volume of zero mode and 2^{g-1} is due to center of the group. This is exactly Verlinde formula for dimension of the space of conformal blocks.

It is necessary to comment on the form of the action (4). We have considered topologically nontrivial bundles over Riemann surface. In this case there is no well defined connection and we need covariant generalization of (4) for this case. It is simple to verify that the following action have all necessary properties.

$$S_{GWZW} = \int_\Sigma g^{-1} \bigtriangledown_A g g^{-1} \bigtriangledown_A + \tag{101}$$
$$\int_B (g^{-1} \bigtriangledown_A g g^{-1} \bigtriangledown_A g g^{-1} \bigtriangledown_A g + F(A)(g \bigtriangledown_A g^{-1} - g^{-1} \bigtriangledown_A g)$$

where $\bigtriangledown_A$ is the covariant derivative and B is the three dimensional manifold which has as the boundary two dimensional surface Σ.

5 Conclusions

We have shown how to get exact answers for correlation functions in GWZW model for compact group. It would be interesting to compare these computations with the possible generalization of Migdal approach [8] to GWZW .It seems that required generalization may be obtained more or less by using quantum finite dimensional groups instead of classical finite dimensional groups. The form of the Verlinde formula seems support this suggestion.

In other direction it would be nice to generalize the calculations described above to the case of non-compact groups connected with theories of gravitational type. Another appealing possibility is two-loop groups closely related with integrable systems [15].

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